UNIVERSITY OF WATERLOO
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ECE 204 *Numerical methods*

Weighted averages

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Introduction

- In this topic, we will
  - Describe a weighted average
  - Look at two applications that use weighted averages to get better approximations of a result

Weighted average

- Given n values, $x_1, \ldots, x_n$, which may be scalars or even vectors, the *arithmetic mean* or just *average* of these is the value

$$\frac{1}{n}\left(x_1 + x_2 + \cdots + x_n\right)$$

- If each of the values $x_1, \ldots, x_n$ approximates some unknown x, then so does the average
  - Under various circumstances, the average may be a better approximation of x than any other sample

Weighted average

- Given n values, $x_1, \ldots, x_n$, which may be scalars or even vectors,
  if $w_1, \ldots, w_n$ are scalars such as
  $$w_1 + w_2 + \cdots + w_n = 1$$
  then a weighted average of these is any linear combination
  $$w_1 x_1 + w_2 x_2 + \cdots + w_n x_n$$

- Assuming each assignment, project, laboratory and examination is graded out of 100, then your final grade in a course is a weighted average of your evaluations
- The arithmetic mean is when
  $$w_1 = w_2 = \cdots = w_n = \frac{1}{n}$$

Weighted average

- There is no requirement that the weights be positive:
  - For example, recall the previous example where we were given n samples uniformly chosen from $[a, b]$, but where we did not know the values of a or b, then one approximation of a was

$$a \approx \min\{x_1, x_2, \ldots, x_n\}$$

  - This was one way to estimate a, but a better way of estimating a was the weighted average:

$$\frac{n \min\{x_1, x_2, \ldots, x_n\} - \max\{x_1, x_2, \ldots, x_n\}}{n-1}$$

    - The weights are $\frac{n}{n-1}$ and $-\frac{1}{n-1}$

No proof of this claim is provided, and not on the examination

Weighted average

- If, however, the weights have the additional property that they are all non-negative, then we are additionally guaranteed that

$$\min\{x_1, x_2, \ldots, x_n\} \le w_1 x_1 + w_2 x_2 + \cdots + w_n x_n \le \max\{x_1, x_2, \ldots, x_n\}$$

- Of course, if all the weights are non-negative, then

$$0 \le w_k \le 1$$

  for all $k = 1, 2, \ldots, n$

- If all the weights are non-negative,
  the weighted average is also called a *convex combination*

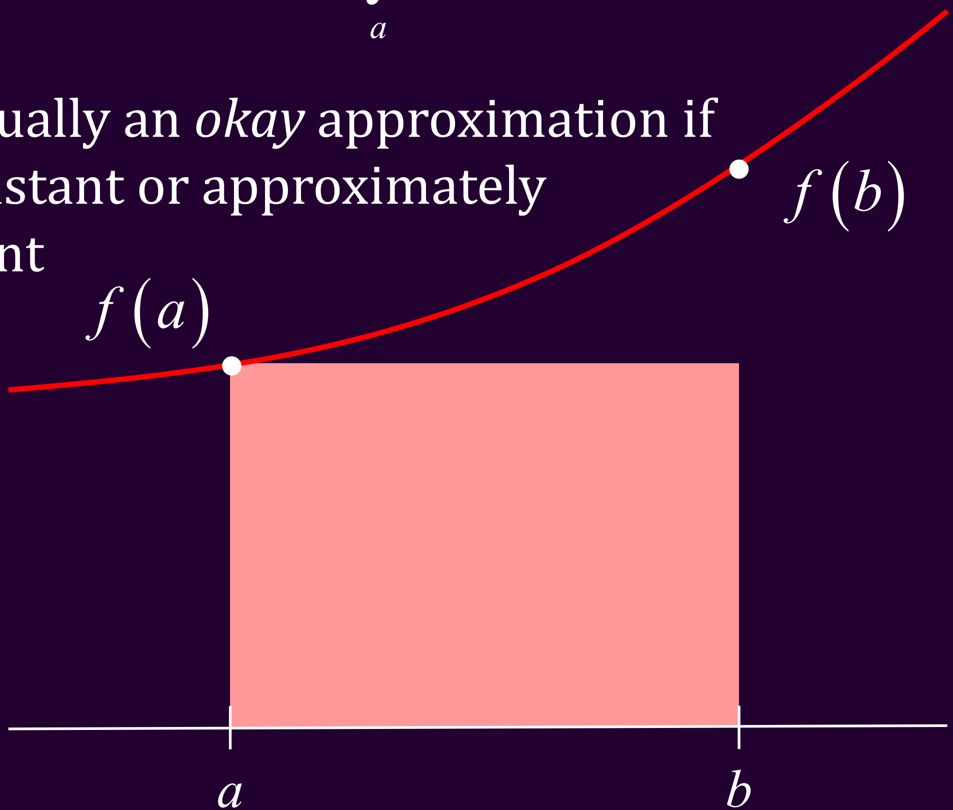
Weighted average

- Important:
  - We will not so much *use* weighted averages, but rather weighted averages will be the result of applying the other tools to finding numerical algorithms

- Bonus:
  - If our algorithm is using a weighted average, it will not be subject to subtractive cancellation
    - This is true even if some of the weights are negative
- We can also use this to catch errors:
  - If a formula appears to be close to a weighted average, but isn't, its probably a good idea to check

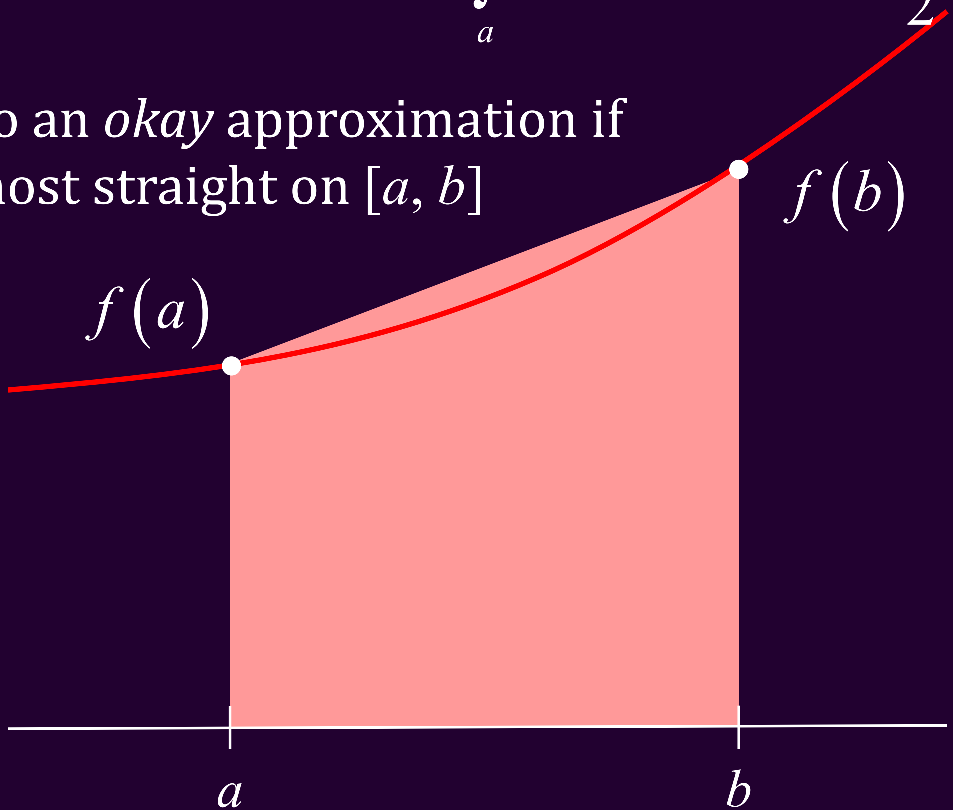
Approximating the integral

- Recall from calculus that $\int_a^b f(x)\,\mathrm{d}x$ is the area under the curve
  - One approximation is $\int_a^b f(x)\,\mathrm{d}x \approx f(a)(b-a)$
  - It's actually an *okay* approximation if f is constant or approximately constant



Approximating the integral

- Recall from calculus that $\int_a^b f(x)\,\mathrm{d}x$ is the area under the curve
  - Another approximation is $\int_a^b f(x)\,\mathrm{d}x \approx \frac{f(a)+f(b)}{2}(b-a)$
  - It's also an *okay* approximation if f is almost straight on $[a, b]$



Approximating the integral

- Now, the point $\frac{a+b}{2}$ is the mid-point of the interval [a, b]
  - Here are two possible approximations:

$$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$$

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{f(a) + f\left(\frac{a+b}{2}\right) + f(b)}{3}(b-a)$$

- This takes the arithmetic mean of the value of the function at three different points

$$\frac{1}{6}, \frac{2}{3}, \frac{1}{6}$$

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)}{6}(b-a)$$

  - This takes a weighted average of those three values

Later (Simpson's rule), we will see why the last formula is superior

Approximating the integral

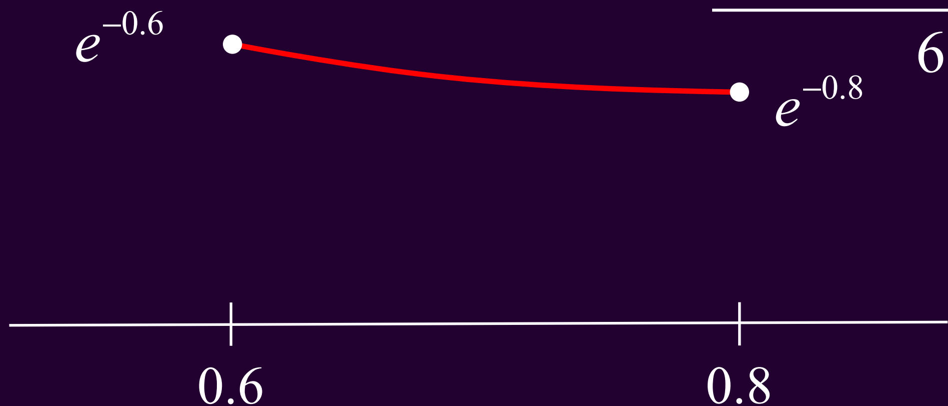
- Examples: $\int_{0.6}^{0,.8} e^{-x}\,\mathrm{d}x \approx 0.09948267197680484 1198$

$$e^{-0.6}0.2 = 0.1097623272188053$$

$$\frac{e^{-0.6}+e^{-0.8}}{2}0.2 = 0.0998140600211248 0$$

$$\frac{e^{-0.6}+e^{-0.7}+e^{-0.8}}{3}0.2 = 0.0996483936001771 7$$

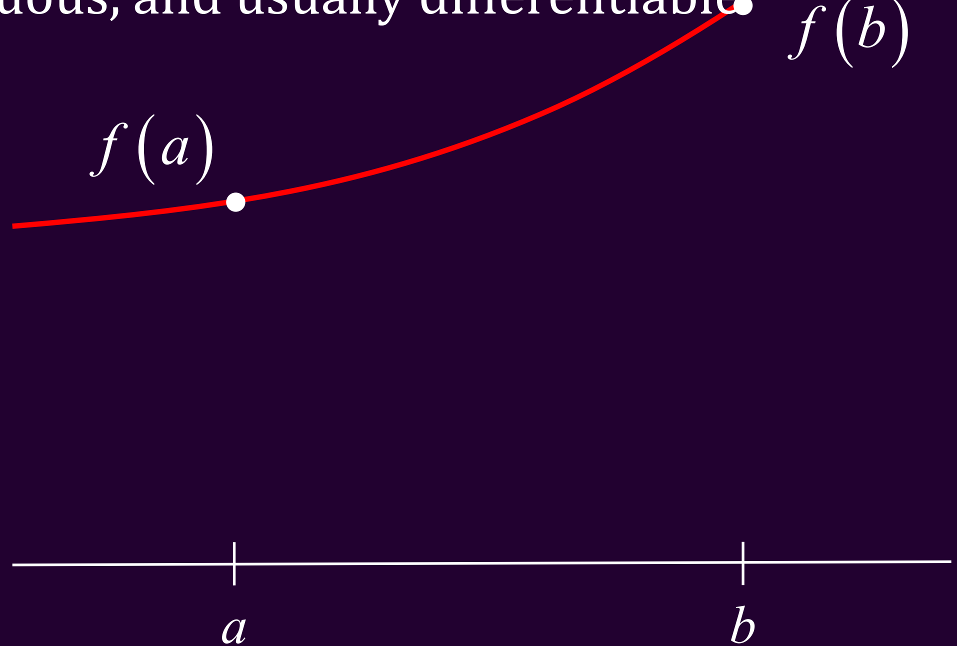
$$\frac{e^{-0.6}+4e^{-0.7}+e^{-0.8}}{6}0.2 = 0.0994827271792295 4$$



Later (Simpson's rule), we will see why the last formula is superior

Approximating the integral

- Question: which is better?
  - If your function is a constant signal with white noise,
    the arithmetic mean is best
  - Engineers, however, deal with the real world, and in the real world, there is momentum, so most functions are nice, continuous, and usually differentiable



Summary

- Following this topic, you now
  - Understand the concept of a weighted average
    - The arithmetic mean is a special case of a weighted mean
  - Know that the weights are not necessarily positive
  - Have seen two applications where weighted averages are used
    - In both applications, a weighted average provided a better approximation than the arithmetic mean

References

[1] https://en.wikipedia.org/wiki/Weighted_arithmetic_mean

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.
Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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